THE RIEMANN SPHERE OF A COMMUTATIVE BANACH ALGEBRA[1]

BY

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Introduction. The Riemann sphere, or extended complex plane C_∞, has long played an important role in classical function theory. Therefore in abstract function theory, where the range or both the domain and range of an analytic function lie in a Banach space or algebra A, it is natural to pose the problem: how should A be extended to a new structure A_∞ which plays the role of the Riemann sphere? Here we give the solution to this problem, valid when A is a complex commutative Banach algebra with identity. A_∞ is provided with quasi-algebraic, topological, and analytic structure.

In §1, the quasi-algebraic structure of A_∞ is studied, with the exposition being given for the surprisingly abstract context of a commutative ring with identity. We say quasi-algebraic rather than algebraic because elements of A_∞ may only sometimes be added and sometimes multiplied. A_∞ may be regarded as the set obtained by adjoining to A all formal quotients a/b of elements of A, where b is singular and both a and b lie in no proper ideal of A. The 2×2 matrices with coefficients in A and invertible determinant induce the fractional linear group of bijections of A_∞. Ring homomorphisms induce Riemann sphere homomorphisms and fractional linear group homomorphisms which are related; this lends a functorial tinge to the subject.

In §2, we equip A_∞ (where A is now required to be a commutative complex B-algebra with unit) with the unique topology so that A is an open subspace and each fractional linear transformation is a homeomorphism. Locally compact Riemann spheres occur iff A is finite dimensional, and compact Riemann spheres are even rarer. The lifting of algebra homomorphisms to Riemann sphere homomorphisms enables the extension of the Gelfand representation of A to A_∞, as well as the definition of the spectrum of an element of A_∞.

The Riemann sphere construction enables us to replace one disconnectedness phenomenon by another. The mapping $a \to a^{-1}$ is usually regarded as being defined on the (perhaps not connected) set I of invertible elements of A. However, here we regard $a \to a^{-1}$ as a mapping of all of A_∞ onto itself, and clearly I is contained in the component of A_∞ which contains A. Unfortunately Example 2.6.5 shows that A_∞ itself need not be connected. A related phenomenon is that the spectrum of an element of A_∞ may be the whole extended plane; whenever A_∞ is disconnected there are such elements, but not conversely.

In §3, we give A_∞ an analytic structure, in which each fractional transformation

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is an analytic automorphism, via the Lorch analytic function theory. We assume only the most basic results of this theory, all of which may be found in Lorch's original paper [6]. Excellent discussions illuminating the nature of this theory and its relation to the other abstract function theories are to be found in [7] and [8]. A fairly complete account of the various analytic function theories in B-spaces and B-algebras is available in Hille and Phillips [5].

The central theorem of §3 is the generalization of the classical theorem that each entire function which approaches infinity at infinity is a complex polynomial. The appropriate formulation of this generalization requires that special attention be paid to the radical of A, as well as to the nontrivial idempotents of A. (If A is semisimple, and has no nontrivial idempotents, then each analytic function on A, with a limit at infinity in an appropriate sense, is a polynomial on A with invertible leading coefficient.) As a corollary, the group of analytic automorphisms of the connected component of A_∞ which contains A is shown, via the classical monodromy theorem, to be the fractional linear group.

REMARK. Although this paper is oriented towards abstract function theory and spectral theory, the Riemann sphere construction is of topological-algebraic interest, since it associates to each commutative (real or complex, although we only consider complex scalars in the paper proper) B-algebra with identity A, a topological space A_∞ which is a topological-algebraic invariant of A, but not a topological invariant. It is interesting to consider the three real Banach algebras obtained by taking the plane R^2, with pointwise linear operations, and the multiplications and norms given by

(1) $(s, t)(u, v)=(su, tv)$, $\|(s, t)\|=\max(|s|, |t|)$,

(2) $(s, t)(u, v)=(su-tv, sv+ut)$, $\|(s, t)\|=(|s|^2+|t|^2)^{1/2}$,

(3) $(s, t)(u, v)=(su, sv+ut)$, $\|(s, t)\|=|s|+|t|$.

In (1) the Riemann sphere is the surface of a torus, in (2) it is the classical Riemann sphere C_∞, and in (3) it is the surface of an open cylinder.

With the exception of 3.4, the results of this paper are contained in the author's Columbia University dissertation [4], which also contains other results bearing on the Lorch theory in general, and on this paper in particular. In a forthcoming paper, we will discuss the applications of the Riemann sphere to operational calculus, with particular reference to the work of Taylor on unbounded closed operators [11].

We should note that de Bruijn [2] has sketched a different definition of Riemann sphere, valid for a not necessarily commutative Banach algebra A. In the commutative case, his sphere is the smallest subset of A_∞ which contains A and is invariant under the complex fractional linear transformations. This notion is not strong enough to permit the definition of such quotients as $\sin a/\cos a$ for singular $\cos a$. De Bruijn's treatment involves neither the ideals of A nor the algebraic structure of A_∞, both of which play an important role here. In [2] the concept of the Riemann sphere as an "analytic A-manifold" is introduced.

I am happy to acknowledge my many invaluable discussions with E. R. Lorch about analytic function theory in Banach algebras, during which he pointed out the need for a theory of fractional linear transformations, and suggested that the next stage in the evolution of function theory in a Banach algebra should be the mixture of classical function-theoretic notions with the ideal theory of the algebra. I would also like to acknowledge several stimulating conversations with R. Bott, and the help of H. Appelgate with Example 2.6.5. The referee has considerably simplified the proof of 3.1.1.

Notation and terminology. *Let S, S_1 and S_2 be sets.*

1. The identity map of S onto itself is denoted by 1_S or $1(S)$.
2. If $f\colon S_1 \to S_2$ and $S \subseteq S_1$, $f|S=$ the restriction of f to S.
3. If $f\colon S \to S_1$ and $g\colon S_1 \to S_2$, then $g \circ f$ denotes the composition of f and g.

Let Y be a topological space.

4. D is a domain in Y iff D is a region in Y iff D is an open connected subset of Y.
5. $C(Y)=$ the Banach algebra of all continuous bounded complex-valued functions on Y with pointwise algebraic operations and the sup norm.
6. $C=$ the complex number field.

Let A be a commutative complex Banach algebra with identity.

7. A is irreducible iff the only idempotents in A are 0 and 1.
8. $\mathscr{M}=$ the maximal ideal space of A. As usual, we identify the maximal ideals M of A with the associated complex algebra homomorphisms $F\colon A \to C$.
9. $\hat{}$ denotes the Gelfand representation of A into $C(\mathscr{M})$, while $\hat{A}$ denotes the image of $\hat{}$.
10. As is usual, we identify the complex number 1 with the identity element of A. Thus C is considered to be a subset of A.
11. z will be used to denote complex numbers and complex variables, while a will denote elements of A and A-variables.
12. If $a_0 \in A$ and R is a nonnegative extended real number, $B(a_0; R)$ denotes the open norm ball of radius R about a_0, while $\bar{B}(a_0; R)$ denotes the closed norm ball of radius R about a_0.
13. If $z_0 \in C$ and R is a nonnegative extended real number, $K(z_0; R)$ denotes the open disc in the complex plane of radius R about z_0, while $\bar{K}(z_0; R)$ denotes the closed disc in the complex plane of radius R about z_0.
14. If A-domains and C-domains are under consideration at the same time, the C-domains will be called complex domains, while the A-domains will simply be called domains. Similar conventions hold with respect to C-holomorphic functions and A-holomorphic functions, etc.
15. In general, the terminology used for classical function theory will be that found in Saks and Zygmund [10]. The terminology used for abstract function theory will parallel that used for classical function theory, except in the introduction.

1. **The Riemann sphere of a commutative ring.** Throughout §1, A will denote a commutative ring with identity.

1.1. *The definition of the Riemann sphere of A.* Two elements s and t of A are said to have no common zero iff there is no proper ideal of A which contains both s and t. Clearly, "proper" may be replaced by "maximal" in the preceding definition without altering the meaning. It is also clear that s and t have no common zero iff there are a, b in A so that $sa+bt=1$.

Note that if $A=C$, s and t have no common zero iff s and t are not both 0. If A is $C[0, 1]$, s and t have no common zero iff there is no x in $[0, 1]$ at which $s(x) = t(x)=0$.

For brevity, we shall often write "(s, t) is admissible" in place of "s and t have no common zero." Let S be the set of admissible ordered pairs (s, t) of elements of A. We introduce a relation R in S by $(s, t)R(u, v)$ iff $sv=tu$. A basic description of R is provided by

THEOREM 1.1.1. *Let (s, t) and (u, v) be admissible. Then $(s, t)R(u, v)$ iff there is an invertible x in A so that $xs=u$ and $xt=v$.*

Proof. One implication is trivial. Now assume that $(s, t)R(u, v)$. Choose a, b in A so that $as+bt=1$. Then $u=uas+but=(au+bv)s$. Similarly $v=(au+bv)t$. If $au+bv$ is singular, then there is some proper ideal I which contains $au+bv$. But then the above equations show that I contains both u and v, which contradicts the admissibility of (u, v). Therefore $x=au+bv$ is invertible; furthermore $u=xs$ and $v=xt$.

That R is an equivalence relation follows immediately from 1.1.1. We now define A_∞ = the Riemann sphere of A, to be the set of all R-equivalence classes of S. If (s, t) is admissible, the equivalence class to which it belongs will be denoted by (s, t).

1.2. *Algebraic operations on the Riemann sphere.* The complex point at infinity is algebraically related to the finite complex numbers and to itself by the formulas $z+\infty=\infty$, $z\cdot\infty=\infty$ for $z\neq 0$, and $\infty\cdot\infty=\infty$. However, $\infty+\infty$ and $0\cdot\infty$ are undefined. We will show that these notions generalize to the ring setting.

Let D_+ be the set of ordered pairs (p, q) of elements of A_∞ so that if we write $p=(s, t)$ and $q=(u, v)$, then (t, v) is admissible. Similarly, let $D^\cdot$ be the set of all (p, q) in $A_\infty \times A_\infty$ so that if $p=(s, t)$ and $q=(u, v)$, then (s, v) and (u, t) are admissible. That these definitions, as well as those we shall make further on, do not depend on the choice of equivalence class representative, is an easy consequence of 1.1.1.

D_+ and $D^\cdot$ are actually the sets of those pairs of elements of A_∞ which can be added and multiplied, respectively. More precisely, we have

THEOREM 1.2.1. *Suppose (s, t) and (u, v) are admissible. Then $((s, t), (u, v))$ is in D_+ iff $(sv+ut, tv)$ is admissible. Similarly, $((s, t), (u, v))$ is in $D^\cdot$ iff (su, tv) is admissible.*

The proof of 1.2.1 involves only elementary algebra, and is thus left to the reader.

1.2.1 enables us to define addition and multiplication on D_+ and $D^\cdot$ respectively as follows: $+ : D_+ \to A_\infty$ is given by $(s, t)+(u, v)=(sv+ut, tv)$, while $\cdot : D^\cdot \to A_\infty$ is defined by $(s, t)\cdot(u, v)=(su, tv)$. We summarize the basic algebraic properties of these operations with

THEOREM 1.2.2. *Let p, q and r be elements in A_∞. Then*

(i) *if $p+q$ $(p\cdot q)$ is defined, then so is $q+p$ $(q\cdot p)$, and $p+q=q+p$ $(p\cdot q=q\cdot p)$,*

(ii) *if $(p+q)+r$ $((p\cdot q)\cdot r)$ is defined, then so is $p+(q+r)$ $(p\cdot(q\cdot r))$, and $(p+q)+r =p+(q+r)$ $((p\cdot q)\cdot r=p\cdot(q\cdot r))$,*

(iii) *$p+(0, 1)$ $(p\cdot(1, 1))$ is defined and $=p$,*

(iv) *if $p\cdot q+p\cdot r$ is defined then so is $p\cdot(q+r)$, and $p\cdot q+p\cdot r=p(q+r)$.*

Proof. We will prove (iv). Write $p=(s, t)$, $q=(u, v)$ and $r=(x, y)$. $p\cdot q+p\cdot r =(sxtv+suty, t^2vy)$, so t must be invertible. (If t lies in some proper ideal I, then so do $sxtv+suty$ and t^2vy, which is impossible, since they have no common zero.) A similar argument shows that v and y have no common zero. Hence by the definition of D_+, $q+r$ is defined and $=(uy+vx, vy)$. That $p\cdot(q+r)$ is defined and $=p\cdot q+p\cdot r$ now follows directly from 1.2.1.

In contrast with addition and multiplication, inverses are everywhere defined on A_∞ by $(s, t)^{-1}=(t, s)$. The following theorem about inverses can be easily proved.

THEOREM 1.2.3. *If p is in A_∞, then $(p^{-1})^{-1}=p$. If p and q are in A_∞, and $p\cdot q$ is defined, then $p^{-1}\cdot q^{-1}$ is defined and $=(p\cdot q)^{-1}$.*

1.3. *The fractional linear transformations.* Unless otherwise specified, all 2×2 matrices will be assumed to have coefficients in A. The matrix $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ will be called regular iff the determinant $eh-gf$ is invertible in A. The regular 2×2 matrices form a group $GL(A)$ under matrix multiplication.

For each $T=\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ in $GL(A)$, define $T' : A_\infty \to A_\infty$ by $T'(s, t)=(es+ft, gs+ht)$. To show that T' is well defined, we must show that $es+ft$ and $gs+ht$ have no common zero. Suppose I is an ideal of A which contains both $es+ft$ and $gs+ht$. Then $ges+gft$ and $egs+eht$ both lie in I; subtracting, we see that $(eh-gf)t$ is in I. Since $eh-gf$ is invertible, t is in I. A similar argument shows that I contains s. Since (s, t) is admissible, $I=A$.

The fractional linear transformations of A are defined to be those mappings of A_∞ into itself which are of the form T', where T is in $GL(A)$. The set of all fractional linear transformations of A will be denoted by $G(A)$. Some fundamental properties of $G(A)$ are given by

THEOREM 1.3.1. *Each fractional linear transformation T' is a bijection of A_∞. $G(A)$ is the subgroup of the group (under composition), of all bijections of A_∞. The mapping $T\to T'$ is a group homomorphism of $GL(A)$ onto $G(A)$ whose kernel consists of those matrices of the form $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, where x is an invertible element of A.*

Proof. We prove the assertion about the kernel. It is clear that if x is invertible, $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)'$ is the identity mapping of A_∞. Now suppose that $T' = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)'$ is the identity mapping. Since $T'(0, 1) = (f, h) = (0, 1)$, $f = 0$ and h is invertible. Since $T'(1, 0) = (e, g) = (1, 0)$, $g = 0$ and e is invertible. Since $T'(1, 1) = (e, h) = (1, 1)$, $e = h$.

We conclude this section with the essential

THEOREM 1.3.2. *$G(A)$ acts transitively on A_∞.*

Proof. It is sufficient to show that for any (s, t) in A_∞, there is some f.l.t. T' which sends $(0, 1)$ into (s, t). Choose a, b in A so that $as + bt = 1$. Clearly $\left(\begin{smallmatrix} -b & s \\ a & t \end{smallmatrix}\right)'(0, 1) = (s, t)$.

1.4. *Mappings induced by ring homomorphisms.* Let A_1 and A_2 be commutative rings with identity, $f\colon A_1 \to A_2$ a ring (not necessarily a ring with identity) homomorphism. Write $j = f(1)$; clearly $j^2 = j$.

LEMMA 1.4.1. *Suppose that (s, t) is A_1-admissible and that y is in A_2. Then $(f(s) + (1-j)y, f(t) + 1 - j)$ is A_2-admissible.*

Proof. Choose a, b in A so that $as + bt = 1$. Then

$$f(a)(f(s) + (1-j)y) + (jf(b) + 1 - j)(f(t) + 1 - j) = 1.$$

1.4.1 shows that the function $f_*\colon A_{1\infty} \to A_{2\infty}$ given by $f_*(s, t) = (f(s), f(t) + 1 - j)$ is well defined. It also follows from 1.4.1 that if (p, q) lies in D_{1+} $(D_1^{\cdot})$, then $(f_*(p), f_*(q))$ is in D_{2+} $(D_2^{\cdot})$. A routine computation now proves

THEOREM 1.4.2. *Let p and q be in $A_{1\infty}$. If $p + q$ $(p \cdot q)$ is defined, then $f_*(p) + f_*(q)$ $(f_*(p) \cdot f_*(q))$ is defined and $= f_*(p + q)$ $(f_*(p \cdot q))$.*

In view of 1.4.2, f_* will be called the Riemann sphere homomorphism induced by f. It is important to note that the assignment of Riemann spheres A_∞ to rings A, and Riemann sphere maps f_* to ring maps f is a covariant functor. In other words, a direct computation shows

THEOREM 1.4.3. *If $f_1\colon A_1 \to A_2$ and $f_2\colon A_2 \to A_3$ are ring homomorphisms, then $(f_2 \circ f_1)_* = f_{2*} \circ f_{1*}$. Furthermore, $(1_A)_* = 1_{A_\infty}$.*

f also induces a group homomorphism from $G(A_1)$ into $G(A_2)$. To define this mapping, we need two lemmas.

LEMMA 1.4.4. *If x is regular in A_1, then $y = f(x) + 1 - j$ is regular in A_2. If $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $GL(A_1)$, then*

$$T_f = \begin{pmatrix} f(a) + 1 - j & f(b) \\ f(c) & f(d) + 1 - j \end{pmatrix}$$

is in $GL(A_2)$.

Proof. A simple computation shows that y^{-1} is given by $f(x^{-1}) + 1 - j$. The second statement of 1.4.3 now follows directly from the first.

LEMMA 1.4.5. *Suppose that T and U are in $GL(A_1)$, and that $T'=U'$. Then $T'_f=U'_f$.*

Proof. By 1.3.1, there is an invertible x in A so that $xT=U$. Therefore

$$(f(x)+1-j)T_f = U_f.$$

1.4.5 now follows from 1.3.1 and 1.4.3.

Now we can define $f^*: G(A_1) \to G(A_2)$ as follows; if T is in $GL(A_1)$, then $f^*(T') = T'_f$. 1.4.3 and 1.4.4 show that f^* is well defined.

It is easy to verify that f^* is a group homomorphism; f^* will be called the group homomorphism induced by f. Again we have a covariant functor, this time assigning groups to rings, and group maps to ring maps. Explicitly, by a direct computation we have

THEOREM 1.4.6. *If $f_1: A_1 \to A_2$ and $f_2: A_2 \to A_3$ are ring homomorphisms, then $(f_2 \circ f_1)^* = f_2^* \circ f_1^*$. Furthermore $(1_A)^* = 1_{G(A)}$.*

The following relationship between f_* and f^*, which can be directly verified, will be of vital importance.

THEOREM 1.4.7. *Suppose X is a fractional linear transformation of A_1. Then*

$$f^*(X) \circ f_* = f_* \circ X \quad \text{and} \quad f^*(X)^{-1} \circ f_* \circ X = f_*.$$

1.5. *The identification of A with a subset of A_∞.* We now identify A with a subset of A_∞ via $a \leftrightarrow (a, 1)$. If a and b are in A, then $a+b$ and ab are independent of whether the A-operations or the A_∞-operations are being considered, as is a^{-1} for invertible a. (From now on, for simplicity of notation, we will write pq instead of $p \cdot q$ to denote the product of two elements of A_∞.) However, if a is a singular element of A, a^{-1} is not defined in A, but is defined in A_∞; there $a^{-1}=(1, a)$. From now on, for a in A, a^{-1} will denote the A_∞-inverse of a. We can thus write

$$A_\infty = \{ab^{-1}; (a, b) \text{ admissible, } a, b \text{ in } A\}.$$

Therefore the extension of A to A_∞ allows us in some sense to divide a by b, whenever a and b are elements of A with no common zero.

Now let A_1, A_2 and f be as in 1.4. Via the above identification, f_* is actually an extension of f to $A_{1\infty}$. If (a, b) is A_1-admissible, then $f_*(ab^{-1}) = f(a)(f(b)+1-j)^{-1}$.

2. **The Riemann sphere of a commutative Banach algebra.** Throughout the remainder of this paper, A will denote a complex commutative Banach algebra with identity. A_∞ will denote the Riemann sphere of (the underlying ring of) A.

2.1. *The definition and elementary properties of the topology for A_∞.* We now introduce a topology for A_∞, by prescribing a neighborhood system $\mathcal{N}_p$ for each point p of A_∞. We define $\mathcal{N}_p$ to be the set of those subsets N of A_∞ which satisfy

(i) If X is a f.l.t. which maps 0 into p, (such X always exist because $G(A)$ is transitive), then there is an open neighborhood U of 0 in A so that $X(U)$ is contained in N.

LEMMA 2.1.1. *Let X be a fractional linear transformation. Suppose that a and $X(a)$ both lie in A, and that V is an open neighborhood of $X(a)$ in A. Then there is an open neighborhood U of a in A so that $X(U)\subseteq V$.*

Proof. Write $X=\begin{pmatrix} e & f \\ g & h \end{pmatrix}'$. Since $X(a)$ is in A, $ga+h$ is invertible. Choose a neighborhood W of a in A so that $gb+h$ is invertible when b lies in W. Clearly $X|W$ is a continuous mapping of A into itself, so U can be chosen to be an appropriate subset of W.

LEMMA 2.1.2. *Suppose that X is a fractional linear transformation, and that V is an open subset of A. If p lies in $X(V)$, then $X(V)$ belongs to $\mathcal{N}_p$.*

Proof. Let Y be a f.l.t. which maps 0 into p. By 2.1.1, choose an open neighborhood U of 0 in A so that $X^{-1}\circ Y$ maps U into V. Clearly $Y(U)$ is contained in $X(V)$, so $X(V)\in\mathcal{N}_p$.

We can now prove

THEOREM 2.1.3. *The assignment $p\to\mathcal{N}_p$ defines a topology for A_∞ in which for each p, $\mathcal{N}_p$ is the set of all neighborhoods of p.*

Proof. All but the last of the standard neighborhood axioms are trivial to verify. Now let $p\in A_\infty$, and $N\in\mathcal{N}_p$. Choose a f.l.t. X which sends 0 into p, and an open neighborhood V of 0 in A so that $X(V)$ is contained in N. By 2.1.2, if q lies in $X(V)$, $X(V)\in\mathcal{N}_q$.

From now on, A_∞ will be considered to be provided with this topology. That this topology is the natural one for A_∞ is made apparent by the following simple theorems.

THEOREM 2.1.4. *If $U\subseteq A$, U is A-open iff U is A_∞-open. Thus A (with the usual norm topology) is an open subspace of A_∞.*

Proof. Let U be A-open, set $X=$ the identity f.l.t. By 2.1.2, $X(U)=U$ is A_∞-open.

Now let U be A_∞-open, let $u\in U$. Set $X=\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}'$, clearly $X(0)=u$. Therefore, there is some $\delta>0$ so that $X(B(0;\delta))\subseteq U$; since $X(B(0;\delta))=B(u;\delta)$, U is A-open.

THEOREM 2.1.5. *Each fractional linear transformation is a homeomorphism of A_∞.*

Proof. It is sufficient to show that each f.l.t. X is continuous at each $p\in A_\infty$. Let N be a neighborhood of $X(p)$. Choose a f.l.t. Y which sends 0 into p, and an open neighborhood U of 0 in A so that $X(Y(U))\subseteq N$. Since $Y(U)$ is a neighborhood of p in A_∞ by 2.1.2, X is continuous at p.

Note that the topology given A_∞ is the unique topology for A_∞ so that A is an open subspace and each f.l.t. is a homeomorphism.

THEOREM 2.1.6. *A_∞ is a Hausdorff space.*

Proof. By 2.1.4 and 2.1.5, it is sufficient to separate 0 from ab^{-1}, where $a, b \in A$ and b is singular. Set $U_1 = B(0; 1)$ and $U_2 = A_\infty \sim \bar{B}(0; 1)$; $\bar{B}(0; 1)$ is A_∞-closed by 2.1.4. Therefore U_1 and U_2 separate 0 and ab^{-1}.

2.2. *Characterizations via the fractional linear group of the complex field and of semisimplicity.* It is well known that the complex fractional linear transformations are 3-point transitive on the classical Riemann sphere. This phenomenon characterizes the complex field as a commutative complex Banach algebra with identity. In fact we have the stronger

THEOREM 2.2.1. *$G(A)$ is 2-point transitive on A_∞ iff A is the complex field.*

Proof. Suppose that 2-point transitivity holds in A_∞. Let a be a nonzero element of A. Choose a f.l.t. X so that $X(a) = 0$ and $X(0) = 1$; write $X = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)'$. Since $X(0) = 1$, h is invertible, and $h = f$. Since $X(a) = 0$, $ga + f$ is invertible and $ea + f = 0$, thus $ga - ea$ is invertible, so a is invertible. Therefore A is a field, but the only complex Banach field is the complex field.

Notice that if $a, b \in A$, and a is invertible, then $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)'$ maps A into itself. Furthermore, if r is in the radical of A, then $\left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)'$ also maps A into itself. Fractional linear transformations of the latter kind are called radical transformations, and will play an important role later on. Fractional linear transformations of the former kind can be used to characterize semisimplicity via

THEOREM 2.2.2. *A is semisimple iff every f.l.t. X which maps A into itself is of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)'$, where a is invertible.*

Proof. Assume that A is semisimple, and that X is a f.l.t. which sends A into itself; write $X = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)'$. Since $X(0) \in A$, h is invertible. Since for each nonzero complex z, $X(z)$ lies in A, $gz + h$ is invertible, so $h^{-1}g$ is in the radical. Since A is semisimple, $h^{-1}g = 0 = g$. Therefore X is of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)'$, where $a = h^{-1}e$ and $b = h^{-1}f$.

Now assume that A is not semisimple; let r be a nonzero element of the radical. $\left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)'$ maps A into itself, and is not of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)'$, by 1.3.1.

2.3. *The elementary topological quasi-algebra of the Riemann sphere.* We begin with an important lemma, which uses the compactness of the maximal ideal space of A.

LEMMA 2.3.1. *Suppose the elements of A a_0 and b_0 have no common zero. Then there is some $\delta > 0$ so that if both $\|a - a_0\|$ and $\|b - b_0\|$ are $< \delta$, then a and b have no common zero.*

Proof. Since a_0 and b_0 have no common zero, there is a positive number 3δ so that $|F(a_0)| + |F(b_0)| \geqq 3\delta$, for all maximal ideals F. If both $\|a - a_0\|$ and $\|b - b_0\|$ are $< \delta$, then $|F(a)| + |F(b)| \geqq \delta$ for all maximal ideals F, so a and b have no common zero.

We will now show that the topology for A_∞ blends nicely with the algebra of A_∞ introduced in §1.

THEOREM 2.3.2. *D_+ is open. $+$ is a continuous mapping of D_+ into A_∞.*

Proof. Let s, t, u and v be elements of A so that (s, t) and (u, v) are admissible, and (st^{-1}, uv^{-1}) lies in D_+. Then (t, v) is admissible. Choose e, f, g and h from A so that $-es+ft=1=-gu+hv$; write $X_1=\left(\begin{smallmatrix} f & s \\ e & t\end{smallmatrix}\right)'$ and $X_2=\left(\begin{smallmatrix} h & u \\ g & v\end{smallmatrix}\right)'$; clearly $X_1(0)=st^{-1}$ and $X_2(0)=uv^{-1}$. By 2.3.1 choose $\delta_1>0$ so that $(ea+t, gb+v)$ is admissible when both a and b lie in $B(0; \delta_1)$. Therefore

$$X_1(B(0; \delta_1))\times X_2(B(0; \delta_1)) \subseteq D_+,$$

so D_+ is open.

Now choose $X_3=\left(\begin{smallmatrix} q & r \\ x & y\end{smallmatrix}\right)'$ so that $X_3(st^{-1}+uv^{-1})=0$. Define ϕ: $B(0; \delta_1)\times B(0; \delta_1) \to A_\infty$ via $\phi(a, b)=X_3(X_1(a)+X_2(b))$. To show that $+$ is continuous on D_+, it is sufficient to show that ϕ is continuous at $(0, 0)$.

For $i=1, \ldots, 4$ define ψ_i: $B(0; \delta_1)\times B(0; \delta_1)\to A$ via

$$\begin{aligned}
\psi_1(a, b) &= (gb+v)(fa+s)+(ea+t)(hb+u),\\
\psi_2(a, b) &= (ea+t)(gb+v),\\
\psi_3(a, b) &= q\psi_1(a, b)+r\psi_2(a, b), \text{ and}\\
\psi_4(a, b) &= x\psi_1(a, b)+y\psi_2(a, b).
\end{aligned}$$

Clearly all the ψ_i are continuous. A direct computation shows that $\phi(a, b)= \psi_3(a, b)\psi_4(a, b)^{-1}$ for (a, b) in $B(0; \delta_1)\times B(0; \delta_1)$. Since $\phi(0, 0)=0$, $\psi_4(0, 0)$ is invertible; choose δ_2 so that $0<\delta_2<\delta_1$ and $\psi_4(a, b)$ is invertible when (a, b) lies in $B(0; \delta_2)\times B(0; \delta_2)$. Thus on $B(0; \delta_2)\times B(0; \delta_2)$, ϕ is the quotient (in A) of two continuous functions; so ϕ is continuous on $B(0; \delta_2)\times B(0; \delta_2)$.

COROLLARY 2.3.3. *Let Y be a topological space, and let $f_1, \ldots, f_n$ be continuous maps of Y into A_∞ so that if $y \in Y$ and $i\neq j$, then $(f_i(y), f_j(y))$ is in D_+. Then $f_1+\cdots+f_n$ is defined and continuous on Y.*

Proof. That $f_1+\cdots+f_n$ is defined (and is independent of the way parentheses are associated in its definition) follows from 1.2.2. (ii) and the remark that (a, bc) is admissible when (a, b) and (a, c) are admissible. The continuity of $f_1+\cdots+f_n$ follows from 2.3.2.

By arguments analogous with those used for $+$, we can prove

THEOREM 2.3.4. *$D^\cdot$ is open, and $\cdot$ is a continuous mapping of $D^\cdot$ into A_∞. If Y is a topological space, and $f_1, \ldots, f_n$ are continuous mappings of Y into A_∞ so that when $i\neq j$, $f_i(y)f_j(y)$ is defined for each y, then $f_1\cdots f_n$ is a well-defined continuous map of Y into A_∞.*

Note that it is trivial to show that $p\to p^{-1}$ is continuous on A_∞, since ${}^{-1}=\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)'$. An interesting application of this, and the preceding ideas, is provided by

THEOREM 2.3.5. *The invertible elements of A are dense in A iff A is dense in A_∞.*

Proof. Assume the invertible elements of A are not dense in A. Choose an open set U in A so that U contains only singular elements. Then $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)'(U)$ is open in A_∞ and does not intersect A.

Now assume A is not dense in A_∞. Choose an open set V in A_∞ which does not meet A. Let p be an element of V, write $p=st^{-1}$, where s and t are elements of A with no common zero. By 2.3.1 there is some $\delta_1>0$ so that (s, b) is admissible when $\|b-t\|<\delta_1$. Define ϕ: $B(t; \delta_1)\to A_\infty$ via $\phi(b)=sb^{-1}$. Since ϕ is continuous, there is some $\delta_2>0$ so that $\phi(b)$ is not in A when $\|b-t\|<\delta_2$. Thus $B(t; \delta_2)$ does not intersect the invertible elements of A.

THEOREM 2.3.6. *Let f: $A_1\to A_2$ be a ring homomorphism of the underlying rings of A_1 and A_2; if f is continuous then so is f_*.*

Proof. Apply the continuity of f_* at 0, and 1.4.6.

COROLLARY 2.3.7. *If A_1 and A_2 are topologically and algebraically isomorphic, then $A_{1\infty}$ and $A_{2\infty}$ are homeomorphic.*

Proof. Let f: $A_1\to A_2$ and g: $A_2\to A_1$ be continuous ring homomorphisms so that $g\circ f=1(A_1)$ and $f\circ g=1(A_2)$. By the covariance of $_*$, $g_*\circ f_*=1(A_{1\infty})$, $f_*\circ g_*=1(A_{2\infty})$, and f_* and g_* are continuous by 2.3.6.

2.4. *Definition and elementary properties of the Gelfand transform and the spectrum of elements of A_∞.* Let F be a maximal ideal of A, i.e. a continuous algebra homomorphism of A onto C. By 2.3.6 F lifts to a continuous "algebraic" homomorphism F_* of A_∞ onto C_∞ = the extended complex plane. Now let p be an element of A_∞. Define $\hat{p}$, the Gelfand transform of p, to be the mapping of $\mathscr{M}$ into C_∞ defined by $\hat{p}(F)=F_*(p)$. This definition clearly agrees with the usual definition when $p\in A$.

To see that $\hat{p}$ is continuous on $\mathscr{M}$, write $p=ab^{-1}$, where a and b are elements of A with no common zero. A direct computation shows that $\hat{p}=\hat{a}(\hat{b})^{-1}$, which is continuous by 2.3.4.

Now define the spectrum of p (written $\sigma(p)$) to be $\hat{p}(\mathscr{M})$. Since $\mathscr{M}$ is compact and p is continuous, $\sigma(p)$ is a compact set in the extended plane. If p lies in A, this notion of spectrum agrees with the usual definition.

Observe that ∞ belongs to the spectrum of p iff p is not in A. A proof runs as follows; obviously if $p\in A$, $\sigma(p)$ is contained in the finite plane. If p is not in A, write $p=ab^{-1}$ as above; b is singular. Let F be a maximal ideal so that $F(b)=0$, clearly $F_*(p)=\infty$.

With this definition of spectrum, and 1.4.7, a routine computation proves the following spectral mapping theorem for complex fractional linear transformations.

THEOREM 2.4.1. *Suppose X is a fractional linear transformation on A which is of the form $\binom{e\ f}{g\ h}'$, where e, f, g and h are complex numbers. Then $\sigma(X(p))=X(\sigma(p))$.*

We will need later not only the continuity of $F\to\hat{p}(F)$ and $p\to\hat{p}(F)$, but the stronger

THEOREM 2.4.2. *The mapping ϕ: $\mathscr{M}\times A_\infty\to C_\infty$ defined by $\phi(F, p)=\hat{p}(F)$ is continuous.*

Proof. We will show continuity at (F_0, p_0). Choose $X=\begin{pmatrix} e & f \\ g & h\end{pmatrix}'$ from $G(A)$ so that $X(0)=p_0$. Define $\psi: \mathscr{M}\times A_\infty \to \mathscr{M}\times A_\infty$ via $\psi(F,p)=(F, X(p))$; since ψ is a homeomorphism and $\psi(F_0, 0)=(F_0, p_0)$ it is sufficient to show that $\phi\circ\psi$ is continuous at $(F_0, 0)$. Since $F_0^*(X)^{-1}$ is a homeomorphism of C_∞ it is sufficient to show that $\theta=F_0^*(X)^{-1}\circ\phi\circ\psi$ is continuous at $(F_0, 0)$.

Define ξ_1 and ξ_2 to be the maps of $\mathscr{M}\times A$ into C given by

$$\xi_1(F, a) = F_0(h)F(ea+f)+F_0(-f)F(ga+h),$$
$$\xi_2(F, a) = F_0(-g)F(ea+f)+F_0(e)F(ga+h).$$

Since $\phi|\mathscr{M}\times A$ is continuous, ξ_1 and ξ_2 are continuous. A direct computation shows that when (F, a) lies in $\mathscr{M}\times A$,

$$\theta(F, a) = \xi_1(F, a)\xi_2(F, a)^{-1}.$$

Thus by 2.3.4, $\theta|\mathscr{M}\times A$ is continuous, so ϕ is continuous at $(F_0, 0)$.

2.5. *Finite products of Riemann spheres and compactness.* Recall that if $A_1,\dots, A_n$ are commutative Banach algebras with identity, their direct sum is defined to be

$$A_1\oplus\cdots\oplus A_n = \{(a_1,\dots, a_n): a_i\in A_i\},$$

where the algebraic operations are pointwise, and

$$\|(a_1,\dots, a_n)\| = \sup \|a_i\|.$$

$A_1\oplus\cdots\oplus A_n$ is also a commutative Banach algebra with identity. We will show that the identity mapping of $A_1\oplus\cdots\oplus A_n$ can be lifted to a natural homeomorphism of the Riemann sphere of $A_1\oplus\cdots\oplus A_n$ with the product of the Riemann spheres of the A_i.

For $1\leq i\leq n$ define $P_j: A_1\oplus\cdots\oplus A_n\to A_j$ and $Q_j: A_j\to A_1\oplus\cdots\oplus A_n$ by

$$P_j(a_1,\dots, a_n) = a_j, \qquad Q_j(a_j) = (0,\dots, a_j,\dots, 0).$$

P_j and Q_j are both continuous ring homomorphisms so that

$$P_j\circ Q_j = 1(A_j), \quad \text{and} \quad P_k\circ Q_j = 0 \quad \text{when } j\neq k.$$

We state without proof the simple and well-known

LEMMA 2.5.1. *If F_j is a maximal ideal of A_j, then $F_j\circ P_j$ is a maximal ideal of $A_1\oplus\cdots\oplus A_n$. Conversely, if F is a maximal ideal of $A_1\oplus\cdots\oplus A_n$, then there is some j, and a maximal ideal F_j of A_j so that $F=F_j\circ P_j$.*

THEOREM 2.5.2. *The mappings $P_\#: (A_1\oplus\cdots\oplus A_n)_\infty\to A_{1\infty}\times\cdots\times A_{n\infty}$ and $Q_\#: A_{1\infty}\times\cdots\times A_{n\infty}\to(A_1\oplus\cdots\oplus A_n)_\infty$ defined below satisfy the conditions*

$P_\#$ and $Q_\#$ are homeomorphisms so that $Q_\#=(P_\#)^{-1}$,

$P_\#|A_1\oplus\cdots\oplus A_n=Q_\#|A_1\oplus\cdots\oplus A_n=1(A_1\oplus\cdots\oplus A_n)$.

Proof. P_j and Q_j lift to continuous "algebraic" maps $P_{j*}: (A_1 \oplus \cdots \oplus A_n)_\infty \to A_{j\infty}$ and $Q_{j*}: A_{j\infty} \to (A_1 \oplus \cdots \oplus A_n)_\infty$ via 1.4.2 and 2.3.6. 1.4.3 shows that $P_{j*} \circ Q_{j*} = 1(A_{j\infty})$. Let π_j be the continuous projection of $A_{1\infty} \times \cdots \times A_{n\infty}$ onto $A_{j\infty}$. We need the following technical lemma.

LEMMA 2.5.3. *If* $i \neq j$ *and* $x \in A_{1\infty} \times \cdots \times A_{n\infty}$, *then* $Q_{i*}(\pi_i(x)) + Q_{j*}(\pi_j(x))$ *is defined in* $(A_1 \oplus \cdots \oplus A_n)_\infty$.

Proof (of 2.5.3). If 2.5.3 is not true, then there is some maximal ideal F of $A_1 \oplus \cdots \oplus A_n$ so that

$$F_*(Q_{i*}(\pi_i(x))) = F_*(Q_{j*}(\pi_j(x))) = \infty.$$

Choose K so that $1 \leqq K \leqq n$, and there is a maximal ideal F_K of A_K so that $F = F_K \circ P_K$; we may assume without loss of generality that $K \neq i$. But now

$$\infty = F_* \circ Q_{i*} \circ \pi_i(x) = F_{K*} \circ P_{K*} \circ Q_{i*} \circ \pi_i(x) = F_{K*} \circ (P_k \circ Q_i)_* \circ \pi_i(x) = 0,$$

which is impossible, so 2.5.3 is proved.

We now return to the proof of 2.5.2. By 2.5.3 and 2.3.4, we can define $Q_\#: A_{1\infty} \times \cdots \times A_{n\infty} \to (A_1 \oplus \cdots \oplus A_n)_\infty$ via

$$Q_\#(x) = \sum_{i=1}^{n} Q_{i*}(\pi_i(x)).$$

$Q_\#$ is continuous on its domain. Now define $P_\#$ to be the mapping of $(A_1 \oplus \cdots \oplus A_n)_\infty$ into $A_{1\infty} \times \cdots \times A_{n\infty}$ given by

$$P_\#(p) = (P_{1*}(p), \ldots, P_{n*}(p)).$$

$P_\#$ is also continuous. But

$$P_\# \circ Q_\#(p_1, \ldots, p_n) = \left(P_{m*}\left(\sum_{i=1}^{n} Q_{i*}(p_i)\right)\right)_{m=1}^{n}$$

$$= \left(\sum_{i=1}^{n} (P_m \circ Q_i)_*(p_i)\right)_{m=1}^{n} = (p_1, \ldots, p_n).$$

Furthermore

$$Q_\# \circ P_\#(ab^{-1}) = Q_\# \circ P_\#((a_1, \ldots, a_n)(b_1, \ldots, b_n)^{-1})$$

$$= Q_\#(a_1 b_1^{-1}, \ldots, a_n b_n^{-1}) = \sum_{i=1}^{n} Q_{i*}(a_i b_i^{-1})$$

$$= \left(\sum_{i=1}^{n} Q_i(a_i)\right)\left(\sum_{i=1}^{n} Q_i(b_i) + 1 - \sum_{i=1}^{n} Q_i(1)\right)^{-1} = ab^{-1}.$$

Thus $P_\# \circ Q_\#$ and $Q_\# \circ P_\#$ are both identity mappings, so $P_\#$ and $Q_\#$ are homeomorphisms and $Q_\# = (P_\#)^{-1}$. It is clear that $P_\#$ and $Q_\#$ leave $A_1 \oplus \cdots \oplus A_n$ fixed.

Evidence of the naturality of $P_{\#}$ and $Q_{\#}$ is provided by

THEOREM 2.5.4. *The functions* $P^{\#}: G(A_1 \oplus \cdots \oplus A_n) \to G(A_1) \times \cdots \times G(A_n)$ *and* $Q^{\#}: G(A_1) \times \cdots \times G(A_n) \to G(A_1 \oplus \cdots \oplus A_n)$ *defined below satisfy the conditions*
$P^{\#}$ and $Q^{\#}$ are group isomorphisms so that $Q^{\#} = (P^{\#})^{-1}$.
If $X \in G(A_1 \oplus \cdots \oplus A_n)$, *then* $P^{\#}(X) \circ P_{\#} = P_{\#} \circ X$.

Proof. Via 1.4, we have the group homomorphisms

$$P_j^*: G(A_1 \oplus \cdots \oplus A_n) \to G(A_j) \quad \text{and} \quad Q_j^*: G(A_j) \to G(A_1 \oplus \cdots \oplus A_n).$$

Define $P^{\#}: G(A_1 \oplus \cdots \oplus A_n) \to G(A_1) \times \cdots \times G(A_n)$ by

$$P^{\#}(X) = (P_1^*(X), \ldots, P_n^*(X)).$$

Define $Q^{\#}: G(A_1) \times \cdots \times G(A_n) \to G(A_1 \oplus \cdots \oplus A_n)$ by

$$Q^{\#}(X_1, \ldots, X_n) = Q_1^*(X_1) \cdots Q_n^*(X_n).$$

Obviously $P^{\#}$ is a group homomorphism. Furthermore, a direct computation shows that $Q^{\#} \circ P^{\#}$ and $P^{\#} \circ Q^{\#}$ are identity maps, thus $Q^{\#}$ is a group isomorphism and $Q^{\#} = (P^{\#})^{-1}$.

Now let X be in $G(A_1 \oplus \cdots \oplus A_n)$. By 1.4.7 $P_j^*(X) \circ P_{j*} = P_{j*} \circ X$. It follows easily that $P^{\#}(X) \circ P_{\#} = P_{\#} \circ X$.

The preceding discussion of products enables us to answer the question: when is A_∞ compact? Note that A_∞ is locally compact iff A is locally compact iff the underlying vector space of A is finite dimensional.

THEOREM 2.5.5. *A_∞ is compact iff A is algebraically isomorphic to the product of n complex planes.*

Proof. Let $f: A \to C^n$ and $g: C^n \to A$ be complex algebra homomorphisms so that $g \circ f = 1(A)$ and $f \circ g = 1(C^n)$. Since A is finite dimensional, f and g are both continuous. Thus f_* and g_* provide a homeomorphism of A_∞ and C_∞^n; but C_∞^n is compact by 2.5.2. (In fact, C_∞^n is homeomorphic to the product of n 2-spheres.) Therefore A_∞ is compact.

Let A_∞ be compact, by the remark preceding the statement of 2.5.5, the underlying vector space of A has finite dimension N. Suppose A has $N+1$ distinct maximal ideals $F_1, \ldots, F_{N+1}$. Set $\mathcal{M}' = \{F_1, \ldots, F_{N+1}\}$, define $\phi: A \to C(\mathcal{M}')$ by $\phi(a) = \hat{a}|\mathcal{M}'$. ϕ is a homeomorphism of A onto $C(\mathcal{M}')$, which contradicts dim $C(\mathcal{M}') \leqq \dim A$. Thus A has exactly n distinct maximal ideals, where $n \leqq N$.

Suppose r is in the radical of A, and $r \neq 0$. Since A_∞ is compact, the sequence mr, $m = 1, 2, \ldots$ has a cluster point p in A_∞. Let F be a maximal ideal, since F_* is continuous, $F_*(p) = 0$. Therefore $\sigma(p) = \{0\}$, so p lies in A, which is clearly impossible. We conclude that A is semisimple, and $\hat{}$ is a topological isomorphism of A onto $C(\mathcal{M}) \approx C^n$.

2.6. *The connected components of the Riemann sphere. Elements of A_∞ whose spectrum is the whole extended plane.* The principal connected component of A_∞ is the connected component of A_∞ which contains A. It will be denoted by $A_{\infty p}$.

Note that all the components of A_∞ are homeomorphic, since $G(A)$ is transitive, and each f.l.t. is a homeomorphism. Since A_∞ is locally connected, each component of A_∞ is open in A_∞.

THEOREM 2.6.1. *Suppose p is an element of A_∞ whose spectrum is a proper subset of the extended plane. Then p belongs to the principal component of A_∞.*

Proof. Choose $z \neq 0$ from C_∞ so that z is not in $\sigma(p)$. Let X be a complex fractional linear transformation which sends 0 into 0 and z into ∞. By 2.4.1 $\infty \notin \sigma(X(p))$, so $X(p)$ lies in A, and therefore in $A_{\infty p}$. Since $X(0)$ and $X(p)$ lie in $A_{\infty p}$, 0 and p belong to the same component of A_∞; thus $p \in A_{\infty p}$.

We now give two examples of elements p of A_∞ whose spectrum is C_∞. In the first example, p lies in $A_{\infty p}$, while in the second p does not. Thus there are algebras whose Riemann sphere is disconnected.

EXAMPLE 2.6.2. Let $A = C$ (the finite complex plane R^2). Define $f, g \colon R^2 \to C$ via $f(z) = z$ when $|z| \leqq 1$, $f(z) = z/|z|$ when $|z| \geqq 1$, $g(z) = 1$ when $|z| \leqq 1$, $g(z) = 1/|z|$ when $|z| \geqq 1$. Clearly f and g are in $C(R^2)$. Let I be an ideal of $C(R^2)$ which contains both f and g, then $h = f\bar{f} + g$ is in I; since $|h(z)| \geqq 1$ for all z, h is invertible, so $I = A$. Therefore f and g have no common zero, set $p = fg^{-1} \in A_\infty$. For $w \in R^2$, let F_w be the maximal ideal of A given by evaluation at w; clearly $F_{w^*}(p) = f(w)g(w)^{-1} = w$. Thus the spectrum of p contains the finite plane; since the spectrum is compact $\sigma(p) = C_\infty$. Now set $X = \begin{pmatrix} g & -f \\ f & 1 \end{pmatrix}' \in G(A)$. $X(p) = 0$ and $X(0) = -f$, so $X(p)$ and $X(0)$ are in $A_{\infty p}$. Thus 0 and p lie in the same component of A_∞; $p \in A_{\infty p}$.

The preceding example can be used to answer negatively a question raised by Blum [1, p. 359]. Blum asks: If $p(a) = x_0 + x_1 a + \cdots + x_n a^n$ is a polynomial with coefficients in a commutative complex Banach algebra with identity A, and $p(a)$ is singular for all a in A, is there a maximal ideal F of A so that $F(x_j) = 0$, all j? Let A, f, g and h be as in 2.6.2; define $p(a) = ga - f$. We have already shown that there is no proper ideal I which contains both f and g; we will show that $p(a)$ is singular for each a in A. Let $a \in A$, choose $r \geqq 0$ so that the range of a is contained in the closed complex disc of radius r. By the Brouwer fixed point theorem, the restriction of a to $\bar{K}(0; r)$ has a fixed point z^*. Thus $p(a)(z^*) = 0$, so $p(a)$ is singular.

We precede the second example with two lemmas.

LEMMA 2.6.3. *Let ϕ be a continuous A_∞-valued function defined on* [0, 1]. *Let a and b be elements of A with no common zero so that $\phi(0) = ab^{-1}$. Then there are two continuous A-valued functions ψ_1 and ψ_2 defined on* [0, 1] *so that*

(i) *If $0 \leqq t \leqq 1$, $\psi_1(t)\psi_2(t)^{-1}$ is defined in A_∞ and $= \phi(t)$, and*

(ii) $\psi_1(0) = a$, $\psi_2(0) = b$.

Proof. That ψ_1 and ψ_2 can be defined in a neighborhood of 0 so as to satisfy

(i) and (ii) follows from writing $\phi(t)=X^{-1}(X(\phi(t)))$, where X is a f.l.t. which sends $\phi(0)$ into 0, and then applying 1.1.1. A standard extension argument now completes the proof.

LEMMA 2.6.4. *Let ψ_1 and ψ_2 be continuous A-valued functions defined on* [0, 1] *so that*

(i) *For each t, $\psi_1(t)\psi_2(t)^{-1}$ is defined in A_∞, and*

(ii) $\psi_1(1)\psi_2(1)^{-1}=0$.

Then ψ_1 and ψ_2 can be extended to continuous A-valued functions (*also called ψ_1 and ψ_2*) *on* [0, 3] *so that* (i) *is satisfied, and* $\psi_1(3)=\psi_2(3)=1$.

Proof. If $1\leqq t\leqq 2$ set $\psi_1(t)=t-1$ and $\psi_2(t)=\psi_2(1)$. If $2\leqq t\leqq 3$ set $\psi_1(t)=1$ and $\psi_2(t)=(t-2)+(3-t)\psi_2(1)$.

EXAMPLE 2.6.5. Set $A=C(S^3)$, where

$$S^3 = \text{the 3-sphere} = \{(z_1, z_2): |z_1|^2+|z_2|^2 = 1,\ z_1, z_2 \text{ complex}\}.$$

Define f and g: $S^3\to C$ via $f(z, w)=z$ and $g(z, w)=w$; f and g are in A. f and g clearly have no common zero; write $p=fg^{-1}\in A_\infty$.

Suppose that p lies in the principal component of A_∞. Then there is a continuous map ϕ of [0, 1] into $A_{\infty p}$ so that $\phi(0)=p$ and $\phi(1)=0$. By 2.6.3 and 2.6.4 choose continuous A-valued functions ψ_1^* and ψ_2^* defined on [0, 1] so that

(i) If $0\leqq t\leqq 1$, $(\psi_1^*(t), \psi_2^*(t))$ is admissible,

(ii) $\psi_1^*(0)=f_0$, $\psi_2^*(0)=g_0$, and

(iii) $\psi_1^*(1)=1=\psi_2^*(1)$.

Define ψ_1, ψ_2: $S^3\times[0, 1]\to C$ by $\psi_i(x, t)=\psi_i^*(t)(x)$; ψ_1 and ψ_2 are continuous. Define ψ_3: $S^3\times[0, 1]\to S^3$ via

$$\psi_3 = (\psi_1/(|\psi_1|^2+|\psi_2|^2)^{1/2}, \psi_2/(|\psi_1|^2+|\psi_2|^2)^{1/2}).$$

$\psi_3(\ \ , 0)$: $S^3\to S^3$ is the identity map, while $\psi_3(\ \ , 1)$: $S^3\to S^3$ maps S^3 onto the point $(1/\sqrt{2}, 1/\sqrt{2})$. Thus the identity map of S^3 is homotopic to the constant map on S^3, contradicting the well-known noncontractability of S^3. Therefore p does not lie in the principal component of A_∞. 2.6.1 shows that the spectrum of p is the extended plane. Furthermore, the Riemann sphere of A is not connected.

3. Abstract analytic function theory.

3.1. *The quotient of a holomorphic function by a maximal ideal.* Suppose that D is a domain in A, f: $D\to A$ is holomorphic, i.e. analytic in the sense of Lorch [6, p. 417], on D, and F is a maximal ideal of A. If there is a (necessarily unique) complex holomorphic function g defined on the complex domain $F(D)$ so that $g\circ F=F\circ f$ on D, we say g is the quotient function of f with respect to F, and write $g=f_F$.

This fundamental concept, introduced in [6] is investigated in detail in [4]. In particular, it is shown there that if D is star shaped, f_F exists. However, an example is given of a simply connected D, a holomorphic f on D, and a maximal ideal F so that f_F does not exist. Here we will present only those basic results required in the sequel.

Notice that if D is a ball $B(0; s)$, f_F is always defined. In fact, if f is given by the power series $f(a)=\sum_n c_n a^n$, f_F is defined on $K(0; s)$ by $f_F(z)=\sum_n F(c_n)z^n$.

It is simple, but important, to notice also that if f and g are holomorphic on A, then $(g \circ f)_F = g_F \circ f_F$. Furthermore $1(A)_F = 1(C)$.

THEOREM 3.1.1. *Let f be holomorphic on A, and let H be the complex entire functions with the compact open topology. Define $\phi\colon \mathscr{M} \to H$ by $\phi(F)=f_F$. Then ϕ is a continuous mapping.*

Proof. Let F_0 be a maximal ideal. To show the continuity of ϕ at F_0, we must show that for any two positive numbers R and ε, there is a neighborhood U of F_0 in $\mathscr{M}$ so that

$$|f_F(z)-f_{F_0}(z)| < \varepsilon,$$

whenever F lies in U and $|z| \leqq R$. Choose $N>0$ so that

$$\sum_{n=N+1}^{\infty} \|c_n\| R^n < \varepsilon/3.$$

Set $U=\{F: |F(c_n)-F_0(c_n)| < \varepsilon/3(n+1)R^n, n=0,\ldots,N\}$. If $F \in U$ and $|z| \leqq R$, then

$$\begin{aligned} |f_F(z)-f_{F_0}(z)| &\leqq \sum_{n>N+1} |F(c_n)|R^n + \sum_{n=0}^{N} |F(c_n)-F_0(c_n)|R^n + \sum_{n>N+1} |F_0(c_n)|R^n \\ &< \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon. \end{aligned}$$

3.2. *The definition and elementary properties of meromorphic functions.* Just as in the case $A=C$, we can speak of an "analytic" function f defined on an open set D in A_∞ and taking values in A_∞ if we use $G(A)$ to define "local coordinates" on A_∞. Such functions will be called meromorphic. A formal definition runs as follows: let D be an open set in A_∞, let f be an A_∞-valued function on D, and p be a point of D. f is meromorphic at p iff there are U, X_1 and X_2 so that

(i) U is an open connected neighborhood of p in D,

(ii) X_1 and X_2 are fractional linear transformations so that $X_1(U)$ and $X_2(f(U))$ are both subsets of A,

(iii) $X_2 \circ f \circ X_1^{-1}\colon X_1(U) \to A$ is holomorphic at $X_1(p)$.

f is meromorphic on D iff it is meromorphic at each point of D.

In connection with the above definition, the following remarks are in order. Meromorphic functions are continuous. If D is a domain in A, and f is A-valued, then f is meromorphic iff it is holomorphic. Each fractional linear transformation is meromorphic on A_∞. The composition of meromorphic functions is again meromorphic.

THEOREM 3.2.1. *Let D be a domain in A, f_1 and f_2 holomorphic functions on D so that $f_1(a)f_2(a)^{-1}$ is defined in A_∞ for each a in D (or equivalently, $f_1(a)$ and $f_2(a)$ have no common zero for each a in D). Then $h(a)=f_1(a)f_2(a)^{-1}$ is meromorphic on D.*

Proof. Let $a_0 \in D$. Choose a f.l.t. $X = \left(\begin{smallmatrix} e & b \\ c & d \end{smallmatrix}\right)'$ so that $X(h(a_0)) = 0$. Since h is continuous by 2.3.4, choose $\delta > 0$ so that $U = B(a_0; \delta) \subseteq D$ and $X(h(U)) \subseteq A$. If $a \in U$,

$$X(h(a)) = (ef_1(a) + bf_2(a))(cf_1(a) + df_2(a))^{-1} \in A.$$

Thus the restriction of $X \circ h$ to U is the quotient in A of holomorphic functions and is thus itself holomorphic. It now follows from the remarks preceding the theorem that h is meromorphic.

We do not know if the global converse of 3.2.1 is valid. However, we can prove the local converse, i.e.

THEOREM 3.2.2. *Let D be a domain in A, h meromorphic on D, a_0 a point of D. Then there is some open neighborhood U of a_0 in D, and holomorphic functions f_1 and f_2 defined on U so that $f_1(a)f_2(a)^{-1}$ is defined and equal to $h(a)$ for each a in U.*

Proof. Choose a f.l.t. X which sends $h(a_0)$ into 0, write $X^{-1} = \left(\begin{smallmatrix} e & b \\ c & d \end{smallmatrix}\right)'$. Let U be an open neighborhood of a_0 in D so that $X(h(U)) \subseteq A$; for $i = 0, 1, 2$ define f_i on U by $f_0(a) = X(h(a))$, $f_1(a) = ef_0(a) + b$, and $f_2(a) = cf_0(a) + d$. All the f_i are clearly holomorphic; if $a \in U$, $f_1(a)f_2(a)^{-1}$ is defined and $= h(a)$.

The preceding results enable us to prove

THEOREM 3.2.3. *Let D be a domain in A, h_1 and h_2 meromorphic functions on D so that $h(a) = h_1(a) + h_2(a)$ is defined in A_∞ for each a in D. Then h is meromorphic on D.*

Proof. Let $a_0 \in D$. By 3.2.2 choose an open neighborhood U of a_0 in D, and holomorphic functions f_1, f_2, g_1 and g_2 on U so that $h_1(a) = f_1(a)f_2(a)^{-1}$ and $h_2(a) = g_1(a)g_2(a)^{-1}$ for each a in U. Then the restriction of h to U is given by

$$(g_2f_1 + f_2g_1)(f_2g_2)^{-1},$$

hence h is meromorphic at a_0 by 3.2.1.

It is immediate from 3.2.4 that the sum, when defined, of finitely many meromorphic functions on a domain D in A is again meromorphic. That the product, when defined, of finitely many meromorphic functions is again meromorphic follows from an argument analogous to that given for sums. The restriction that D lie in A, rather than in A_∞, can be easily removed. Finally, observe that since the operation of taking inverses is given by a fractional linear transformation, the inverse of a meromorphic function is always defined and meromorphic.

THEOREM 3.2.4 (IDENTITY THEOREM). *Let D be a domain in A_∞, h_1 and h_2 meromorphic on D. If h_1 and h_2 agree on an open subset of D, then h_1 and h_2 agree on D.*

Proof. By a standard connectedness argument, the observation that locally a meromorphic function can be written $X_2 \circ f \circ X_1$, where X_1 and X_2 are f.l.t.'s and f is holomorphic, and the identity theorem for holomorphic functions.

3.3. *Simple analytic polynomials and simple algebraic polynomials.* A basic theorem of classical function theory is that a complex entire function $f(z)$ is a polynomial iff $\lim_{z\to\infty} f(z)$ exists in C_∞. Our task here is that of generalizing this theorem to the Lorch analytic function theory. With this in mind we formulate the following definitions:

An analytic polynomial (an.p.) on A is a holomorphic function $f: A \to A$ so that $\lim_{z\to\infty, z\in C} f(z)$ (or briefly, $\lim_{z\to\infty} f(z)$) exists in A_∞.

A simple analytic polynomial (s.an.p.) on A is an analytic polynomial f so that if $f(a)=\sum_n c_n a^n$ is the power series expansion of f on A, there is some $K>0$ so that c_K is invertible, and c_n lies in the radical for $n>K$. K is called the degree of f.

In this subsection, we will describe the simple analytic polynomials on A. In 3.5 we will describe the analytic polynomials on A.

Let S denote the class of all A_∞-valued functions defined on A_∞. Let S_Y denote the class of all those functions f in S so that $pf(p)$ is defined for all p in A_∞. Define $Y: S_Y \to S$ by $Y(f)(p)=pf(p)$.

For each a in A, define T_a (translation by a) to be the fractional linear transformation $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)'$. Define $\bar{T}_a: S\to S$ by $\bar{T}_a(f)=T_a\circ f$.

For each r in the radical of A, define R_r (the radical transformation induced by r) by $R_r=\left(\begin{smallmatrix}1 & 0\\ r & 1\end{smallmatrix}\right)'$. Define $\bar{R}_r: S\to S$ by $\bar{R}_r(f)=R_r\circ f$.

For each a in A and r in the radical of A, define $Z(a,r): S_Y\to S$ by $Z(a,r)(f) =\bar{R}_r\circ\bar{T}_a\circ Y(f)$.

LEMMA 3.3.1. *If f lies in S_Y, so does $Z(a,r)(f)$.*

Proof. Let $p\in A_\infty$, write $p=a_1b_1^{-1}$, $f(p)=a_2b_2^{-1}$, where for $i=1,2$ (a_i,b_i) is an admissible pair. Then

$$(\bar{R}_r\circ\bar{T}_a\circ Y(f))(p) = (a_1a_2+ab_1b_2)(b_1b_2+ra_1a_2+rab_1b_2)^{-1}.$$

We must verify that

$$q = (a_1b_1^{-1}, (a_1a_2+ab_1b_2)(b_1b_2+ra_1a_2+rab_1b_2)^{-1})\in D^{\cdot}.$$

Suppose that F is a maximal ideal, and

$$F(a_1a_2+ab_1b_2) = F(b_1) = 0.$$

Then $F(a_1a_2)=0$, since a_1 and b_1 have no common zero $F(a_2)$ is 0, contradicting $(p,f(p))\in D^{\cdot}$. Thus $a_1a_2+ab_1b_2$ and b_1 have no common zero. Now suppose that F is a maximal ideal so that

$$F(b_1b_2+ra_1a_2+rab_1b_2) = F(a_1) = 0.$$

Since r is in the radical, $F(b_1b_2)=0$; since (a_1,b_1) is admissible $F(b_2)=0$, contradicting $(p,f(p))\in D^{\cdot}$. Therefore a_1 and $b_1b_2+ra_1a_2+rab_1b_2$ have no common zero, hence q lies in $D^{\cdot}$.

THEOREM 3.3.2. *Let x_0 be an invertible element of A, let $a_1, \ldots, a_K$ be elements of A, and let $r_1, \ldots, r_K$ be elements of the radical of A. Define f_0 on A_∞ by $f_0(A_\infty) = x_0$; clearly f_0 lies in S_Y. For $1 \leqq j \leqq K$ write $Z_j = Z(a_j, r_j)$. Set*

$$f = Z_K \circ Z_{K-1} \circ \cdots \circ Z_1(f_0).$$

Then f is defined and meromorphic on A_∞, and f maps A into itself. Furthermore, the restriction of f to A is a simple analytic polynomial of degree K.

Proof. 3.3.1 shows that f is defined on A_∞; that f is meromorphic on A_∞ follows from the elementary properties of meromorphic functions. It is clear that f maps A into itself.

We will show that f is a s.an.p. of degree K by induction on K. The case $K=0$ is trivial. For $K>0$, set $f_1 = Z_{K-1} \circ \cdots \circ Z_1(f_0)$, and let $\sum d_n a^n$ be the power series expansion of $f_1|A$. By the inductive hypothesis d_{K-1} is invertible, and d_n is in the radical for $n > K-1$. Define $f_2\colon A \to A$ by $f_2(a) = af_1(a)$. Obviously f_2 is a s.an.p. of degree K. It is also easy to see that f_3 is a s.an.p. of degree K, where f_3 is defined on A by $f_3(a) = f_2(a) + a_K$. Finally, define $f_4\colon A \to A$ by

$$f_4(a) = f_3(a)(r_K f_3(a) + 1)^{-1}.$$

If F is a maximal ideal, $F(f_4(a)) = F(f_3(a))$, thus $f_{4F} = f_{3F}$. Therefore $f|A = f_4$ is a s.an.p. of degree K.

Any meromorphic function f on A_∞ which can be constructed from an invertible x_0, arbitrary $a_1, \ldots, a_K$, and $r_1, \ldots, r_K$ in the radical in the way described in 3.3.2, will be called a simple algebraic polynomial (s.al.p.) of degree K. Under this terminology 3.3.2 says that the restriction of a simple algebraic polynomial of degree K to A is a simple analytic polynomial of degree K.

Note that any polynomial (in the usual sense) on A with invertible leading coefficient can be extended to an s.al.p. Conversely, if A is semisimple and f is an s.al.p., then the restriction of f to A is a polynomial on A with invertible leading coefficient. Thus the construction of a simple algebraic polynomial on A is a generalization of the construction of a complex polynomial, as well as a way of producing simple analytic polynomials.

THEOREM 3.3.3 (UNIQUENESS OF THE CONSTRUCTION OF A S.AL.P.). *Let f be the s.al.p. constructed from $x_0, a_1, \ldots, a_K$ and $r_1, \ldots, r_K$, and let g be the s.al.p. constructed from $x_0', a_1', \ldots, a_J'$ and $r_1', \ldots, r_J'$. If $f|A = g|A$, then $K=J$, $x_0 = x_0'$, $a_j = a_j'$ and $r_j = r_j'$ for all j.*

Proof. Since $f|A$ and $g|A$ are s.an.p.s of degree K and J respectively, $K=J$. We now proceed by induction on K. The case $K=0$ is trivial. Now suppose $K>0$, write $Z_j = Z(a_j, r_j)$ and $Z_j' = Z(a_j', r_j')$. Set

$$f_1 = Z_{K-1} \circ \cdots \circ Z_1(f_0) \quad \text{and} \quad g_1 = Z_{K-1}' \circ \cdots \circ Z_1'(g_0).$$

Let ∞ denote $(1,0)\in A_\infty$. Then (writing $a_j=a(j)$ and $r_j=r(j)$) $f(\infty)=R_{r(K)}\circ T_{a(K)}(\infty f_1(\infty))=R_{r(K)}\circ T_{a(K)}(\infty)=R_{r(K)}(\infty)=r_K^{-1}$. Similarly $g(\infty)=(r'_K)^{-1}$, since $f(\infty)=g(\infty)$, $r_K=r'_K$. Furthermore,

$$R_{-r(K)}(f(0)) = T_{a(K)}(0f_1(0)) = T_{a(K)}(0) = a_K.$$

Applying the same argument to $g(0)=f(0)$, we find that $a_K=a'_K$. Therefore $af_1(a)=ag_1(a)$ for all invertible a, so $f_1=g_1$ on A. An application of the inductive hypothesis completes the proof.

We can now prove the important

THEOREM 3.3.4. *Let g be a simple analytic polynomial. Then there is a unique simple algebraic polynomial f whose restriction to A is g.*

Proof. Let $\sum c_n a^n$ be the power series expansion of g on A. c_K is invertible, where $K=$degree g, and c_n lies in the radical for $n>K$. Set $q=\lim_{z\to\infty} g(z)$. We proceed by induction on K. If $K=0$, and F is a maximal ideal, $F_*(q)=\lim_{z\to\infty} F(g(z))=F(c_0)$. Thus q lies in A; by Liouville's theorem g is identically q.

Now consider $K>0$. If F is a maximal ideal,

$$F_*(q) = \lim_{z\to\infty} F(c_0)+F(c_1)z+\cdots+F(c_K)z^K = \infty.$$

Thus the spectrum of q contains only the point at infinity, so we can write $q=s^{-1}$, where s is in the radical of A.

Set $b=g(0)(1-sg(0))^{-1}$, consider the function g_1 on A defined by $g_1=T_{-b}\circ R_{-s}\circ g$. Clearly g_1 is holomorphic on A, $g_1(0)=0$, and $\lim_{z\to\infty} g_1(z)=\infty$. It is not hard to see that if $\sum_{n=1}^\infty d_n a^n$ is the power series expansion of g_1 on A, and F is a maximal ideal, then $F(c_n)=F(d_n)$, $n\geqq 1$. Thus g_1 is an s.an.p. of degree K.

Now define g_2 on A by $g_2(a)=\sum_{n=0}^\infty d_{n+1}a^n$. Since $\lim_{z\to\infty} g_1(z)=\infty$,

$$\lim_{z\to 0} (g_1(1/z))^{-1} = 0,$$

so $\lim_{z\to 0} (g_1(1/z))^{-1}/z$ exists in A. Therefore $\lim_{z\to\infty} zg_1(z)^{-1}$ exists in A, so by the continuity of ${}^{-1}$ $\lim_{z\to\infty} g_1(z)/z=g_2(z)$ exists in A_∞. Therefore g_2 is a s.an.p. of degree $K-1$. By the inductive hypothesis there is a s.al.p. f_2 so that $f_2|A=g_2$. Now clearly $f=Z(b,s)(f_2)$ is a s.al.p. whose restriction to A is g. The uniqueness of f follows from 3.3.3.

3.4. *The meromorphic automorphisms of the principal component of the Riemann sphere.* As an application of 3.3.4, we will show that each meromorphic automorphism of $A_{\infty p}$ is given by a fractional linear transformation.

LEMMA 3.4.1. *Let $\mathscr{Y}$: $[0,1]\to C_\infty$ be a curve, let $p_0\in A_\infty$. Suppose F is a maximal ideal which sends p_0 into $\mathscr{Y}(0)$. Then there is a curve γ: $[0,1]\to A_\infty$ so that $\gamma(0)=p_0$ and $F_*\circ\gamma=\mathscr{Y}$.*

Proof. If range $\mathscr{Y}$ lies in C, we can define γ by

$$\gamma(t) = p_0+\mathscr{Y}(t)-F_*(p_0).$$

If range $\mathscr{Y}$ is a proper subset of C_∞, choose a complex f.l.t. X so that $\mathscr{Y}'=X\circ\mathscr{Y}$ is a curve in C. Since $F_*(X(p_0))=\mathscr{Y}'(0)$, there is a curve γ': $[0, 1]\to A_\infty$ so that $\gamma'(0)=X(p_0)$ and $F_*\circ\gamma'=\mathscr{Y}'$. Set $\gamma=X^{-1}\circ\gamma'$; $\gamma(0)=p_0$, and

$$F_*\circ\gamma = F_*\circ X^{-1}\circ\gamma' = X^{-1}\circ F_*\circ\gamma' = \mathscr{Y}.$$

If range $\mathscr{Y}=C_\infty$, γ can be constructed by applying the preceding construction on suitable subintervals of [0, 1].

LEMMA 3.4.2. *Let g be meromorphic on $A_{\infty p}$, let F be a maximal ideal. Then there is a (necessarily unique) complex meromorphic function g_F on C_∞ so that $g_F\circ F_*=F_*\circ g\cdot g_F$ is called the quotient function of g with respect to f.*

Proof. Let p be in $A_{\infty p}$. Choose fractional linear transformations X_1 and X_2 so that $X_1(0)=p$ and $X_2(g(p))=0$. Now choose $t>0$ so that $f=(X_2\circ g\circ X_1)|B(0; t)$ is holomorphic. Let f_F: $K(0; t)\to C$ be the quotient function of f by F.

Set $U=X_1(B(0; t))$; U is an open connected neighborhood of p in A_∞. Note that $F_*(U)=F^*(X_1)(K(0; t))$ is an open connected neighborhood of $F_*(p)$ in C_∞. Define g_{F_p}: $F_*(U)\to C_\infty$ by

$$g_{F_p} = F^*(X_2^{-1})\circ f_F\circ F^*(X_1^{-1}).$$

If $q\in U$, two applications of 1.4.7 show that

$$g_{F_p}\circ F_*(q) = F_*\circ g(q). \tag{1}$$

Set $\phi_{F_p}=$the complex analytic element $\{g_{F_p}, F_*(p)\}$ (recall that our notions of complex analytic element, continuation, etc. are taken from Saks and Zygmund [10]). (1) shows that ϕ_{F_p} is determined uniquely by F and p, i.e. is independent of the choice of X_1, X_2 and t. Set

$$\mathscr{F} = \{\phi_{F_p} : p\in A_{\infty p}\}.$$

Observe that if γ: $[0, 1]\to A_{\infty p}$ is a curve in $A_{\infty p}$ it follows from (1) that $\{\phi_{F\gamma(t)}\}_{0\le t\le 1}$ is a complex analytic chain along $F_*\circ\gamma$. Thus any two elements of $\mathscr{F}$ are continuations of each other.

Now let $q\in A_{\infty p}$, and let $\mathscr{Y}$: $[0, 1]\to C_\infty$ be a curve starting at $F_*(q)$. By 3.4.1 let γ: $[0, 1]\to A_\infty$ be a curve starting at q so that $F_*\circ\gamma=\mathscr{Y}$. $\{\phi_{F\gamma(t)}\}_{0\le t\le 1}$ is a continuation of ϕ_{F_q} along γ consisting of elements of $\mathscr{F}$. Thus $\mathscr{F}$ is a complex analytic function, arbitrarily continuable in C_∞.

By the monodromy theorem, $\mathscr{F}$ defines a complex meromorphic function g_F: $C_\infty\to C_\infty$. If $p\in A_{\infty p}$,

$$g_F\circ F_*(p) = g_{F_p}\circ F_*(p) = F_*\circ g(p).$$

So $g_F\circ F_*=F_*\circ g$ on $A_{\infty p}$.

THEOREM 3.4.3. *Let g be a meromorphic automorphism of $A_{\infty p}$, i.e., g is a meromorphic bijection of $A_{\infty p}$ whose inverse is meromorphic. Then there is a unique fractional linear transformation X whose restriction to $A_{\infty p}$ is g.*

Proof. Since $G(A)$ is transitive, we may assume without loss of generality that $g(\infty)=\infty$. Let h be the inverse function of g; for each maximal ideal F form the quotient functions g_F and h_F. If $z \in C$,

$$g_F \circ h_F(z) = g_F \circ h_F \circ F_*(z) = F_* \circ g \circ h(z) = z.$$

Similarly $h_F \circ g_F(z)=z$, so g_F is a complex meromorphic automorphism of C_∞ whose inverse is h_F.

But

$$g_F(\infty) = g_F \circ F_*(\infty) = F_* \circ g(\infty) = \infty.$$

Hence $g_F(z) \neq \infty$ if z is a finite complex number. So if $a \in A$, $g_F \circ F_*(a)=F_* \circ g(a)$ is finite, which implies that g maps A into A. Let $\sum c_n a^n$ be the power series for g on A, then $\sum F(c_n)z^n$ is the power series for g_F on C. Since g_F is a complex f.l.t. which leaves ∞ fixed, $F(c_1)\neq 0$ and $F(c_n)=0$ for $n>1$. Thus $g|A$ is a simple analytic polynomial of degree 1.

By 3.3.4 there is a simple algebraic polynomial f of degree 1 whose restriction to A is g. But any s.al.p. of degree 1 is a f.l.t. X. Since $X=g$ on A, $X=g$ on $A_{\infty p}$.

The uniqueness of X follows from the remark that if X_1 is a f.l.t. which leaves 0, 1 and ∞ fixed, X_1 must be the identity transformation.

3.5. *Analytic polynomials and algebraic polynomials.* In 3.3 it was shown that any simple analytic polynomial is the restriction to A of a simple algebraic polynomial. We will define an algebraic polynomial to be a finite direct sum (in an appropriate sense) of simple algebraic polynomials. Then we will show that each analytic polynomial can be thought of as a "finite direct sum" of simple analytic polynomials, and thus is the restriction to A of an algebraic polynomial.

LEMMA 3.5.1. *Let $A_1, \ldots, A_n$ be commutative complex Banach algebras with identity, $h_1, \ldots, h_n$ meromorphic functions on $A_{1\infty}, \ldots, A_{n\infty}$. Define $h_*: A_{1\infty} \times \cdots \times A_{n\infty} \to A_{1\infty} \times \cdots \times A_{n\infty}$ by*

$$h_*(p_1, \ldots, p_n) = (h_1(p_1), \ldots, h_n(p_n)).$$

Define $h^*: (A_1 \oplus \cdots \oplus A_n)_\infty \to (A_1 \oplus \cdots \oplus A_n)_\infty$ by (see 2.5.2 for the definitions of $P_\#$ and $Q_\#$)

$$h^*(p) = Q_\# \circ h_* \circ P_\#(p).$$

h^* is meromorphic on $(A_1 \oplus \cdots \oplus A_n)_\infty$.

Proof. Let $p \in (A_1 \oplus \cdots \oplus A_n)_\infty$, we will show that h^* is meromorphic at p. Set $P_\#(p)=(p_1, \ldots, p_n)$. For each integer j so that $1 \leqq j \leqq n$ choose

(1) $X_{j1} \in G(A_j)$ so that $X_{j1}(p_j)=0$,

(2) $X_{j2} \in G(A_j)$ so that $X_{j2}(h_j(p_j))=0$,

(3) $\delta_j>0$ so that $X_{j2} \circ h_j \circ X_{j1}^{-1}(B(0; \delta_j)) \subseteq A_j$,

and set $\delta_0=\inf_{1 \leq j \leq n} \delta_j$. Thus $X_{j2} \circ h_j \circ X_{j1}^{-1}$ is holomorphic on $B(0; \delta_j)$; let

$\sum_{k=0}^{\infty} c_{jk}(a_j)^k$ be its power series expansion there. Set (see 2.5.4 for the definition of $Q^{\#}$)

$$X_1 = Q^*(X_{11}, \ldots, X_{n1}) \quad \text{and} \quad X_2 = Q^*(X_{12}, \ldots, X_{n2});$$

X_1 and X_2 lie in $G(A_1 \oplus \cdots \oplus A_n)$. Set $U = X_1^{-1}(B(0; \delta_0))$.

Clearly U is open and connected, and X_1 maps U into A. We will show that $p \in U$ by showing that $X_1(p) = 0$.

$$\begin{aligned} X_1(p) &= Q^{\#}(X_{11}, \ldots, X_{n1}) \circ Q_{\#} \circ P_{\#}(p) \\ &= (\text{by } 2.5.4)\ Q_{\#}(X_{11}, \ldots, X_{n1}) \circ P_{\#}(p) = 0. \end{aligned}$$

Now let $a = (a_1, \ldots, a_n) \in B(0; \delta_0)$. Then

$$\begin{aligned} X_2 \circ h^* \circ X_1^{-1}(a) &= X_2 \circ h^* \circ Q^{\#}(X_{11}^{-1}, \ldots, X_{n1}^{-1}) \circ Q_{\#}(a_1, \ldots, a_n) \\ &= (\text{by } 2.5.4)\ X_2 \circ h^* \circ Q_{\#} \circ (X_{11}^{-1}, \ldots, X_{n1}^{-1})(a_1, \ldots, a_n) \\ &= X_2 \circ Q_{\#} \circ h_* \circ P_{\#} \circ Q_{\#}(X_{11}^{-1}(a_1), \ldots, X_{n1}^{-1}(a_n)) \\ &= Q^{\#}(X_{12}, \ldots, X_{n2}) \circ Q_{\#}(h_1 \circ X_{11}^{-1}(a_1), \ldots, h_n \circ X_{n1}^{-1}(a_n)) \\ &= (\text{since } X_{j2} \circ h_j \circ X_{j1}^{-1} \text{ lies in } A_j \text{ for all } j) \end{aligned}$$

$$(X_{12} \circ h_1 \circ X_{11}^{-1}(a_1), \ldots, X_{n2} \circ h_n \circ X_{n1}^{-1}(a_n)) = \sum_{k=0}^{\infty} c_k a^k,$$

where $c_k = (c_{1k}, \ldots, c_{nk})$. Thus $X_2 \circ h^* \circ X_1^{-1}$ is represented on $B(0; \delta_0)$ by the convergent power series $\sum c_k a^k$, so $X_2 \circ h^* \circ X_1^{-1}$ is holomorphic at 0. Therefore h^* is meromorphic at p. Q.E.D.

Note that if in 3.5.1 we require that each h_j map A_j into itself for all j, then h^* maps A into itself. In fact, when $a = (a_1, \ldots, a_n) \in A_1 \oplus \cdots \oplus A_n$, $h^*(a) = (h_1(a_1), \ldots, h_n(a_n))$.

LEMMA 3.5.2. *Let A and A' be commutative complex Banach algebras with identity. Suppose that $f: A \to A'$ and $f': A' \to A$ are continuous algebra with identity homomorphisms so that $f' \circ f = 1_A$ and $f \circ f' = 1_{A'}$. Let h be meromorphic on A_∞, define $h': A'_\infty \to A'_\infty$ by $h' = f_* \circ h \circ f'_*$. Then h' is meromorphic on A'_∞. Furthermore, if h maps A into itself, then h' maps A' into itself.*

Proof. Let $p' \in A'_\infty$, set $p = f'_*(p')$, and choose X_1 and X_2 from $G(A)$ so that $X_1(p) = 0 = X_2(h(p))$. Choose $\delta > 0$ so that $X_2 \circ h \circ X_1^{-1}$ maps $B(0; \delta)$ into A; $X_2 \circ h \circ X_1^{-1}$ is thus holomorphic on $B(0; \delta)$ with power series expansion $\sum c_n a^n$. Set

$$X_1' = f^*(X_1), \qquad X_2' = f^*(X_2), \quad \text{and} \quad U' = X_1'^{-1}(B(0; \delta/\|f'\|)).$$

Since

$$X_1'(p') = f^*(X_1) \circ f_* \circ f'_*(p') = (\text{by } 1.4.7) f_* \circ X_1(p) = 0,$$

U' is an open connected neighborhood of p; clearly X_1' maps U' into A'. Furthermore when $b' \in A'$ and $\|b'\| < \delta/\|f'\|$, then

$$\begin{aligned} X_2' \circ h' \circ X_1'^{-1}(b') &= X_2' \circ h' \circ f^*(X_1^{-1}) \circ f_* \circ f_*'(b') \\ &= X_2' \circ h' \circ f_* \circ X_1^{-1} \circ f'(b') = f^*(X_2) \circ f_* \circ h \circ f_*' \circ f_* \circ X_1^{-1} \circ f'(b') \\ &= f_* \circ X_2 \circ h \circ X_1^{-1} \circ f'(b') = f_*(\sum c_n(f'(b'))^n) = \sum f(c_n)(b')^n. \end{aligned}$$

Thus $X_2' \circ h' \circ X_1'^{-1}$ is represented by a convergent power series on $B(0; \delta/\|f'\|) = X_1'(U')$. Therefore h' is meromorphic at p'.

Furthermore, if $h(A) \subseteq A$ and $b' \in A'$, then $h'(b') = f_* \circ h \circ f_*'(b') = f(h(f'(b'))) \in A'$. Q.E.D.

We can now define the concept of "algebraic polynomial in A". By a partition of the identity for A we mean a finite sequence $j_1, \ldots, j_n$ of distinct nonzero idempotents so that

$$j_k j_i = 0 \quad \text{when } k \neq i, \text{ and } \sum_{k=1}^{n} j_k = 1.$$

Let $j_i, \ldots, j_n$ be a partition of the identity in A. For $1 \leqq i \leqq n$ let A_i be the commutative Banach algebra with identity obtained by renorming the closed subalgebra $j_i A$ via its regular representation; thus if $\|| \quad \||$ denotes the norm in A_i, $\|| \quad \||$ is equivalent to the norm in A_i induced by A, and $\||j_i\|| = 1$. Let $q_1 \colon A_1 \oplus \cdots \oplus A_n \to A$ and $q_2 \colon A \to A_1 \oplus \cdots \oplus A_n$ be defined by

$$q_1(a_1, \ldots, a_n) = a_1 + \cdots + a_n, \qquad q_2(a) = (j_1 a, \ldots, j_n a).$$

It is not hard to see that q_1 and q_2 are continuous algebra with identity homomorphisms so that q_2 is the inverse mapping of q_1.

A function $\xi \colon A_\infty \to A_\infty$ is an algebraic polynomial (al.p.) on A iff there are $j_1, \ldots, j_n$ and $h_1, \ldots, h_n$ so that

(1) $j_1, \ldots, j_n$ is a partition of the identity in A,

(2) if for each i, $1 \leqq i \leqq n$, we let A_i be the commutative B-algebra with identity obtained by renorming $j_i A$ via the regular representation. Then each h_i is a mapping of $A_{i\infty}$ into itself which is either an A_i simple algebraic polynomial or an A-valued constant mapping, and

(3) if h^* is the meromorphic function on $(A_1 \oplus \cdots \oplus A_n)_\infty$ constructed from $h_1, \ldots, h_n$ as in 3.5.1, and q_1 and q_2 are the isomorphisms of $A_1 \oplus \cdots \oplus A_n$ and A introduced above, then $\xi = q_{1*} \circ h^* \circ q_{2*}$.

3.5.2 and the note preceding 3.5.2 show that each algebraic polynomial is a meromorphic function defined on A_∞ which maps A into A. Thus the restriction to A of an algebraic polynomial is an analytic polynomial.

Theorem 3.5.3. *If g is an analytic polynomial, then g is the restriction to A of an algebraic polynomial.*

Proof. Let $p_0=\lim_{z\leqq\infty} g(z)$, and let $\sum c_n a^n$ be the power series expansion of g on A. If F is a maximal ideal,

$$F_*(p_0) = F_*\Big(\lim_{z\to\infty} g(z)\Big) = \lim_{z\to\infty} g_F(z).$$

Since $\lim_{z\to\infty} g_F(z)$ exists, g_F cannot have an essential singularity at ∞. For each maximal ideal F, set $n(F)=0$ if g_F is constant, and set $n(F)=$ the degree of g_F if g_F is a nonconstant polynomial. We will show that $n(F)$ is continuous on $\mathscr{M}$, i.e., $n(F)$ is constant in a neighborhood of each maximal ideal. Fix a maximal ideal F' of A.

If $n(F')=0$, then $F'_*(p_0)=\lim_{z\to\infty} g_{F'}(z)$ is finite. By the continuity of $\hat{p}_0$, choose a neighborhood V' of F' so that

$$\lim_{z\to\infty} g_F(z) = F_*(p_0)$$

is finite when F is in V_1. Thus $n(F)=0$ on V_1.

Now suppose $n(F')>0$. Then

$$F'_*(p_0) = \lim_{z\to\infty} g_{F'}(z) = \infty.$$

By 2.4.2 choose a neighborhood U of p_0 in A, and a neighborhood N_0 of F' in $\mathscr{M}$ so that either $\hat{p}(F)=\infty$ or $\hat{p}(F)$ is finite and $|\hat{p}(F)|\geqq 1$ when p is in U and F is in N_0. Since $\lim_{z\to\infty} g(z)=p_0$, choose $s>0$ so that $g(z)\in U$ when $|z|\geqq s$. Thus $|g_F(z)| = |F(g(z))|\geqq 1$, if $|z|\geqq s$ and $F\in N_0$. By the continuity of $F\to g_F$ choose a neighborhood N_1 of F' so that $N_1\subseteq N_0$ and

$$|g_F(z)-g_{F'}(z)| \leqq \tfrac{1}{2} \quad \text{when } F\in N_1 \text{ and } |z| = s.$$

For F in N_1 it follows from Rouché's theorem that $g_{F'}$ and g_F have the same number of zeros on $K(0; s)$; thus $g_{F'}$ and g_F have the same number of zeros on C, i.e. $n(F')$ zeros. So for F in N_1, g_F is a polynomial of degree $n(F')$; $n(F)=n(F')$. Thus $n(F)$ is continuous on $\mathscr{M}$.

For each nonnegative integer n, set $\mathscr{M}_n=\{F:n(F)=n\}$. Each $\mathscr{M}_n$ is open and closed in $\mathscr{M}$, and $\mathscr{M}$ is the disjoint union of all the $\mathscr{M}_n$. Since $\mathscr{M}$ is compact, there is some $N>0$ so that $\mathscr{M}_N$ is nonempty, and $\mathscr{M}_n$ is empty for $n>N$. For $0\leqq n\leqq N$ let j'_n be the idempotent of A so that (Rickart [9, p. 168])

$$(j_n'^{\wedge})^{-1}(1) = \mathscr{M}_n.$$

Let $j_0,\dots,j_M$ be the sequence obtained from $j'_0,\dots,j'_N$ by omitting all the zero idempotents; $j_0,\dots,j_M$ is a partition of the identity and $j_M=j'_N$. For $0\leqq i\leqq M$ let A_i be the commutative Banach algebra with identity obtained by renorming j_iA via the regular representation; let $P'_i\colon A\to A_i$ be the continuous homomorphism defined by $P'_i(a)=j_ia$. Define $g_i\colon A_i\to A_i$ by

$$g_i(a_i) = \sum_{n=0}^{\infty} j_ic_n(a_i)^n.$$

Then

$$p_{0i} = P'_{i\bullet}(p_0) = P'_{i\bullet}\left(\lim_{z\to\infty} g(z)\right)$$
$$= \lim_{z\to\infty} P'_i(g(z)) = \lim_{z\to\infty} g_i(z).$$

Thus g_i is an analytic polynomial on A_i. For $0 \leqq i \leqq M$ let $u(i)$ be the integer satisfying $j'_{u(i)} = j_i$.

Suppose $u(i) > 0$. Let G_i be a maximal ideal of A_i, set F_i = the maximal ideal of A given by $G_i \circ P'_i$. If $a \in A$, $F_i(a) = G_i(j_i a)$. Since $F_i(j'_{u(i)}) = G_i(j_i) = 1$, $n(F_i) = u(i)$. So if $n > u(i)$, $G_i(j_i c_n) = F_i(c_n) = 0$, while $G_i(j_i c_{u(i)}) = F_i(c_{u(i)}) \neq 0$. Therefore g_i is a simple analytic polynomial on A_i. By 3.3.4 let h_i be the simple algebraic polynomial on A_i whose restriction to A_i is g_i.

Suppose $u(i) = 0$. By the above argument, we see that if $n > 0$, then $G_i(j_i c_n) = 0$. Therefore

$$G_{i\bullet}(p_{0i}) = \lim_{z\to\infty} G_i(g_i(z)) = G_i(j_i c_0) \in C.$$

Thus ∞ is not in the spectrum of p_{0i} in $A_{i\infty}$, so p_{0i} lies in A_i. By Liouville's theorem g_i is identically constant on A_i. Let h_i be the constant function on $A_{i\infty}$ whose restriction to A_i is g_i.

Use the $h_0, \dots, h_M$ to construct the meromorphic function h^* on $(A_0 \oplus \cdots \oplus A_M)_\infty$ via 3.5.1. Define the algebraic polynomial $\xi: A_\infty \to A_\infty$ via $\xi = q_{1\bullet} \circ h^* \circ q_{2\bullet}$, where q_1 and q_2 are defined as on p via $j_0, \dots, j_M$. If $a \in A$, then

$$\begin{aligned}\xi(a) &= q_1(h_0(j_0 a), \dots, h_M(j_M a)) \\ &= q_1\left(\sum_n (j_0 c_n)(j_0 a)^n, \dots, \sum_n (j_M c_n)(j_M a)^n\right) \\ &= \sum_{i=0}^{M} \sum_{n=0}^{\infty} (j_i c_n)(j_i a)^n = \sum_{i=0}^{M} j_i \sum_{n=0}^{\infty} c_n a^n = g(a).\end{aligned}$$

Thus g is the restriction to A of the generalized polynomial ξ. Q.E.D.

Note that the j'_n of the preceding proof can be constructed directly; i.e. without invoking the Šilov theorem that for any clopen subset $\mathscr{M}_1$ of $\mathscr{M}$ there is an idempotent j in A so that $\hat{j}^{-1}(1) = \mathscr{M}_1$. (The details of the construction appear in the author's dissertation.) Since the proof of the Šilov theorem depends on the theory of functions of several complex variables, we thus avoid having a portion of the theory of analytic functions of one algebra variable depend on the theory of analytic functions of several complex variables.

We now present some illustrative examples of the concepts of simple analytic polynomial and analytic polynomial.

Example 3.5.4. Suppose r is a nonnilpotent element of the radical, define $g: A \to A$ by $g(a) = a(1 - ra)^{-1}$. g is a simple analytic polynomial, but g is not a polynomial.

EXAMPLE 3.5.5. Let $A=C[0, 1]$, define $x_1\colon [0, 1]\to C$ by $x_1(t)=t$, define $g\colon A\to A$ by $g(a)=x_1a$. g is a polynomial but g is neither a simple analytic polynomial nor an analytic polynomial.

EXAMPLE 3.5.6. Let $A=C\times C$, with pointwise operations and the sup norm. Set $x_1=(1, 0)$ and $x_0=(0, 1)$, define $g\colon A\to A$ by $g(a)=x_0+x_1a$. g is an analytic polynomial, but g is not a simple analytic polynomial.

EXAMPLE 3.5.7. Let A be the algebra of bounded sequences of complex numbers, with pointwise operations and the sup norm. For $n\geqq 0$ set $x_n=(\delta_{nk}/k!)_{k=0}^{\infty}$, where δ_{nk} is the Kronecker δ. Define $g\colon A\to A$ by $g(a)=\sum x_na^n$. It follows from 3.5.3 that g is not an analytic polynomial.

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